# Graph Theory 2

Aleksandra Gorzkowska

## Definition

A walk in a graph $G = (V, E)$ is a sequence $v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_k$, where $v_i$ are vertices, $e_i$ are edges and $v_{i-1} v_i = e_i \in E$ for every $i \in \{1, 2, \ldots, k\}$.

It is an alternating sequence of vertices and edges, with consecutive vertices separated by the edge joining them.

## Definition

A walk in a graph $G = (V, E)$ is a sequence $v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$, where $v_i$ are vertices, $e_i$ are edges and $v_{i-1} v_i = e_i \in E$ for every $i \in \{1, 2, \dots, k\}$.

It is an alternating sequence of vertices and edges, with consecutive vertices separated by the edge joining them.

For short, we write $v_0 v_1 v_2 \dots v_k$.

## Definition

A walk in a graph $G = (V, E)$ is a sequence $v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$, where $v_i$ are vertices, $e_i$ are edges and $v_{i-1} v_i = e_i \in E$ for every $i \in \{1, 2, \dots, k\}$.

It is an alternating sequence of vertices and edges, with consecutive vertices separated by the edge joining them.

For short, we write $v_0 v_1 v_2 \dots v_k$.

The number of edges in a walk, namely $k$, is called its length.

## Definition

A walk in a graph $G = (V, E)$ is a sequence $v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$, where $v_i$ are vertices, $e_i$ are edges and $v_{i-1} v_i = e_i \in E$ for every $i \in \{1, 2, \dots, k\}$.

It is an alternating sequence of vertices and edges, with consecutive vertices separated by the edge joining them.

For short, we write $v_0 v_1 v_2 \dots v_k$.

The number of edges in a walk, namely $k$, is called its length.

- If $v_0 = v_k$, the walk is closed.

## Definition

A walk in a graph $G = (V, E)$ is a sequence $v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$, where $v_i$ are vertices, $e_i$ are edges and $v_{i-1} v_i = e_i \in E$ for every $i \in \{1, 2, \dots, k\}$.

It is an alternating sequence of vertices and edges, with consecutive vertices separated by the edge joining them.

For short, we write $v_0 v_1 v_2 \dots v_k$.

The number of edges in a walk, namely $k$, is called its length.

- If $v_0 = v_k$, the walk is closed.
- If all vertices are distinct, the walk is a path.

## Definition

A walk in a graph $G = (V, E)$ is a sequence $v_0 e_1 v_1 e_2 v_2 \dots v_{k-1} e_k v_k$, where $v_i$ are vertices, $e_i$ are edges and $v_{i-1} v_i = e_i \in E$ for every $i \in \{1, 2, \dots, k\}$.

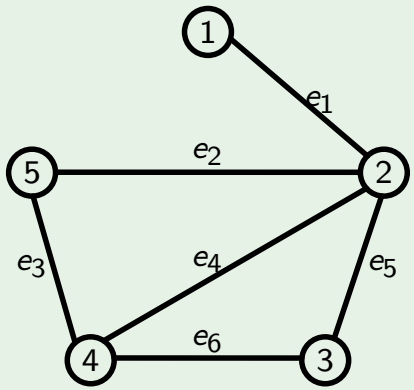
It is an alternating sequence of vertices and edges, with consecutive vertices separated by the edge joining them.

For short, we write $v_0 v_1 v_2 \dots v_k$.

The number of edges in a walk, namely $k$, is called its length.

- If $v_0 = v_k$, the walk is closed.
- If all vertices are distinct, the walk is a path.
- A closed walk with $k \geq 3$ and pairwise distinct vertices $v_1, \dots, v_k$ is a cycle.

## Example




graph $G$

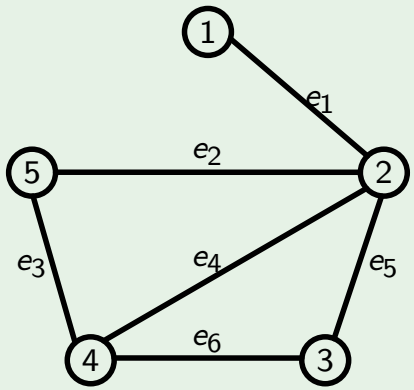
$2e_4 4e_3 5e_2 2e_5 3$ - walk

$2e_4 4e_6 3e_5 2e_2 5e_3 4e_4 2$ - closed walk

$1e_1 2e_5 3e_6 4$ - path

$2e_2 5e_3 4e_6 3e_5 2$ - cycle

## Example




graph $G$

$2e_4 4e_3 5e_2 2e_5 3$ - walk

$2e_4 4e_6 3e_5 2e_2 5e_3 4e_4 2$ - closed walk

$1e_1 2e_5 3e_6 4$ - path

$2e_2 5e_3 4e_6 3e_5 2$ - cycle

The vertices $v_0$ and $v_k$ of a walk/path $v_0 v_1 \dots v_k$ are its endpoints. The walk/path joins these vertices.

## Definition

A graph $G$ is connected if every pair of vertices $x, y \in V(G)$ is joined by a path in $G$.

## Definition

A graph $G$ is connected if every pair of vertices $x, y \in V(G)$ is joined by a path in $G$.

## Definition

A maximal (under inclusion) connected subgraph of $G$ is a component of $G$.

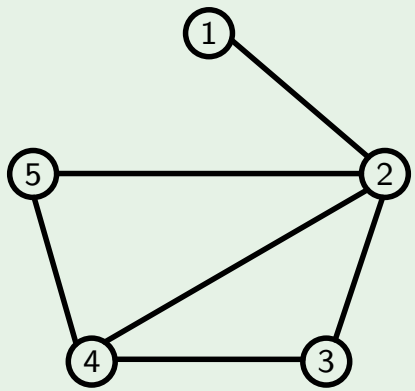
## Definition

A graph $G$ is connected if every pair of vertices $x, y \in V(G)$ is joined by a path in $G$.

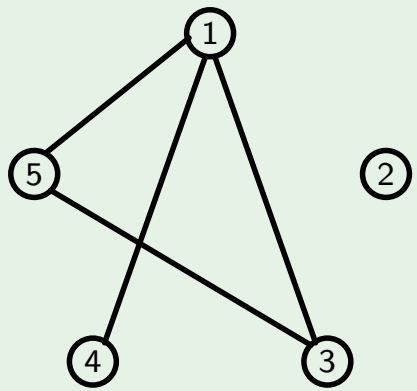
## Definition

A maximal (under inclusion) connected subgraph of $G$ is a component of $G$.

## Example




graph $G$ is connected




graph $\overline{G}$ is NOT connected

Two components:
vertex 2;
the subgraph of $\overline{G}$ induced by $\{1, 3, 4, 5\}$

## Definition

A bridge of a connected graph $G$ is an edge whose deletion disconnects $G$.

**Definition**

A bridge of a connected graph $G$ is an edge whose deletion disconnects $G$.

**Definition**

A cut vertex of a connected graph $G$ is a vertex whose deletion disconnects $G$.

## Definition

A closed walk containing every edge of $G$ exactly once is an Euler tour (also called an Euler circuit).

A graph admitting an Euler tour is Eulerian.

**Definition**

A closed walk containing every edge of $G$ exactly once is an Euler tour (also called an Euler circuit).

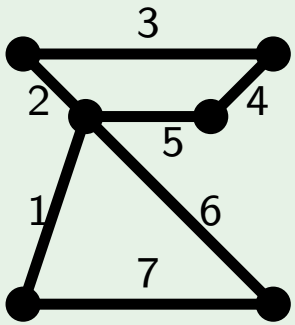
A graph admitting an Euler tour is Eulerian.

**Definition**

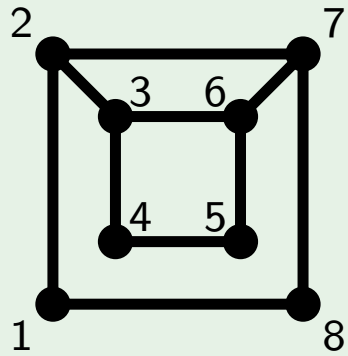
A cycle containing every vertex of $G$ is a Hamilton cycle.

A graph admitting a Hamilton cycle is Hamiltonian.

## Example




Eulerian graph




Hamiltonian graph

**Theorem (Euler, 1760)**

*A connected graph $G$ has an Eulerian cycle if and only if the degree of every vertex is even.*

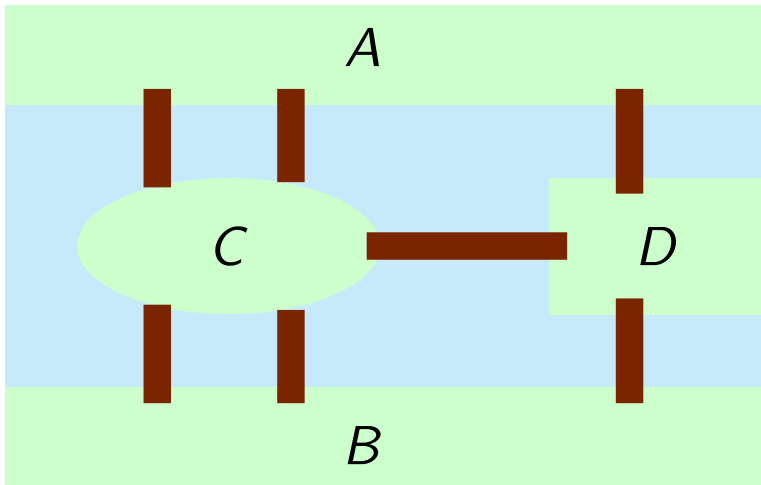
**Theorem (Euler, 1760)**

*A connected graph $G$ has an Eulerian cycle if and only if the degree of every vertex is even.*

This is what we call a necessary and sufficient condition for a graph to be Eulerian.

# The Seven Bridges of Königsberg

Can we take a closed walk that crosses each of the seven bridges exactly once?



# The Chinese Postman Problem

## Graph theory formulation

Given a connected graph with nonnegative edge weights, find a minimum-weight *closed walk* that traverses every edge at least once.

Interpretation: a postman must cover every street and return to the post office. Edge weights represent distances or travel times.

# The Chinese Postman Problem

## Graph theory formulation

Given a connected graph with nonnegative edge weights, find a minimum-weight *closed walk* that traverses every edge at least once.

Interpretation: a postman must cover every street and return to the post office. Edge weights represent distances or travel times.

- If every vertex has even degree, an Euler tour is optimal: each edge is traversed exactly once.

# The Chinese Postman Problem

## Graph theory formulation

Given a connected graph with nonnegative edge weights, find a minimum-weight *closed walk* that traverses every edge at least once.

Interpretation: a postman must cover every street and return to the post office. Edge weights represent distances or travel times.

- If every vertex has even degree, an Euler tour is optimal: each edge is traversed exactly once.
- Otherwise, some edges must be traversed more than once. The goal is to find a closed walk that has smallest possible sum of weights on the edges traversed more than once.

# Necessary Conditions for Hamiltonicity

**Theorem**

*A Hamiltonian graph has no cut vertex.*

# Necessary Conditions for Hamiltonicity

**Theorem**

*A Hamiltonian graph has no cut vertex.*

**Remark**! The converse is not true.

# Necessary Conditions for Hamiltonicity

**Theorem**

*A Hamiltonian graph has no cut vertex.*

**Remark**! The converse is not true.

**Theorem**

*If $G$ is Hamiltonian and $\emptyset \neq S \subset V(G)$, then deleting $S$ from $G$ leaves at most $|S|$ components.*

# Necessary Conditions for Hamiltonicity

**Theorem**

*A Hamiltonian graph has no cut vertex.*

**Remark**! The converse is not true.

**Theorem**

*If $G$ is Hamiltonian and $\emptyset \neq S \subset V(G)$, then deleting $S$ from $G$ leaves at most $|S|$ components.*

**Remark**! The converse is not true.

# Sufficient Conditions for Hamiltonicity

**Theorem**

*Let $G = (V, E)$ be a simple graph of order $n$. If $u, v \in V$ are nonadjacent, $d(u) + d(v) \geq n$, and $G + uv$ is Hamiltonian, then $G$ is Hamiltonian.*

# Sufficient Conditions for Hamiltonicity

## Theorem

*Let $G = (V, E)$ be a simple graph of order $n$. If $u, v \in V$ are nonadjacent, $d(u) + d(v) \geq n$, and $G + uv$ is Hamiltonian, then $G$ is Hamiltonian.*

## Definition

The Bondy–Chvátal $k$-closure of $G$ is obtained by repeatedly joining nonadjacent vertices whose degree sum is at least $k$, until no such pair remains.

Notation: $cl_k(G)$

# Sufficient Conditions for Hamiltonicity

**Theorem**

*Let $G = (V, E)$ be a simple graph of order $n$. If $u, v \in V$ are nonadjacent, $d(u) + d(v) \geq n$, and $G + uv$ is Hamiltonian, then $G$ is Hamiltonian.*

**Theorem**

*If $G$ has order $n$ and $cl_n(G)$ is Hamiltonian (in particular, if $cl_n(G) = K_n$), then $G$ is Hamiltonian.*

# Sufficient Conditions for Hamiltonicity

**Theorem (Ore)**

*If $G = (V, E)$ has order $n \geq 3$ and $d(u) + d(v) \geq n$ for every pair of distinct nonadjacent vertices, then $G$ is Hamiltonian.*

# Sufficient Conditions for Hamiltonicity

**Theorem (Ore)**

*If $G = (V, E)$ has order $n \geq 3$ and $d(u) + d(v) \geq n$ for every pair of distinct nonadjacent vertices, then $G$ is Hamiltonian.*

**Theorem (Dirac)**

*If $G$ has order $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$, then $G$ is Hamiltonian.*

# Sufficient Conditions for Hamiltonicity

The hypotheses of these theorems are called the conditions of Ore and Dirac.

# Sufficient Conditions for Hamiltonicity

The hypotheses of these theorems are called the conditions of Ore and Dirac.

Let $G = (V, E)$ have order $n$. We say that $G$ satisfies:

# Sufficient Conditions for Hamiltonicity

The hypotheses of these theorems are called the conditions of Ore and Dirac.

Let $G = (V, E)$ have order $n$. We say that $G$ satisfies:

Ore's condition if: $\forall u, v \in V : uv \notin E \Rightarrow d(u) + d(v) \geq n$

# Sufficient Conditions for Hamiltonicity

The hypotheses of these theorems are called the conditions of Ore and Dirac.

Let $G = (V, E)$ have order $n$. We say that $G$ satisfies:

Ore's condition if: $\forall u, v \in V : uv \notin E \Rightarrow d(u) + d(v) \geq n$

Dirac's condition if: $\delta(G) \geq \frac{n}{2}$

# The Travelling Salesman Problem (TSP)

## A minimum-cost tour

Given a complete graph with nonnegative edge weights, find a minimum-weight *Hamilton cycle*.

Interpretation: vertices represent cities; edge weights are distances or travel times between cities. A salesman has to visit every city.

# The Travelling Salesman Problem (TSP)

## A minimum-cost tour

Given a complete graph with nonnegative edge weights, find a minimum-weight *Hamilton cycle*.

Interpretation: vertices represent cities; edge weights are distances or travel times between cities. A salesman has to visit every city.

The goal is to find a Hamilton cycle that has smallest possible sum of weights on the edges traversed.

# The Travelling Salesman Problem (TSP)

## A minimum-cost tour

Given a complete graph with nonnegative edge weights, find a minimum-weight *Hamilton cycle*.

Interpretation: vertices represent cities; edge weights are distances or travel times between cities. A salesman has to visit every city.

The goal is to find a Hamilton cycle that has smallest possible sum of weights on the edges traversed.

TSP is NP-hard; choosing the nearest unvisited city at each step does not guarantee an optimal tour.